Massachusetts Institute of Technology
1.200J—Transportation Systems Analysis: Performance and Optimization
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Recitation 7
Unit 3 — Probabilistic Methodology and Examples of Applications

More Queues (for One Last Time!)

1 M/·/1 Queues

Consider a single-server queueing system with infinite queue capacity. Arrivals of customers to this system occur in a Poisson manner at the rate of 27 per hour. The service times, S, of customers are mutually independent. Compute the expected waiting time in queue, T_q, and the expected number of customers in the queue, N_q, when this queueing system is in steady state.

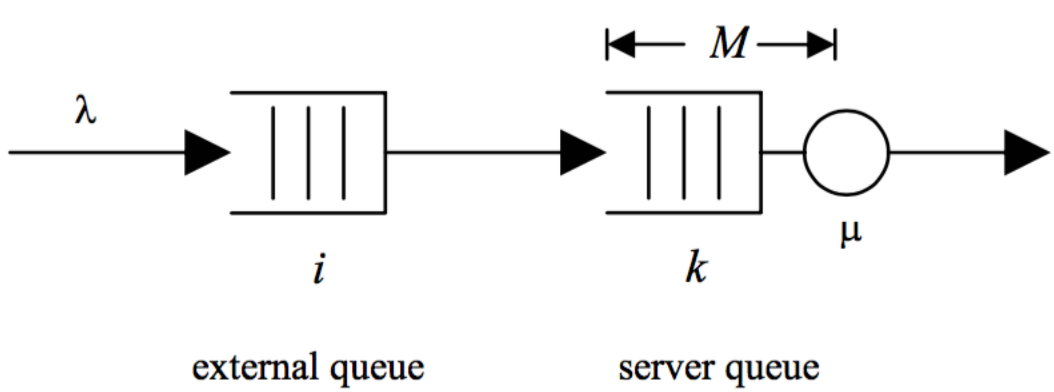
(a) S has a negative exponential probability density function with $f_(s)(t) = 0.5e^{-0.5t}$ for $t \geq 0$, in units of minutes. [For M/M/1 with arrival rate λ and service rate μ, the average total time in the system is $1/(\mu - \lambda)$]

(b) S is uniformly distributed between 1 and 3 minutes, in other words, $f_s(t) = U[1, 3]$ in units of minutes. [Variance of uniform $[x, y]$ is $((y - x)^2)/12$.]

(c) S is constant and has duration equal to 2 minutes.

2 Two Queues

Consider an M/M/1 queue that can accommodate at most M customers in the system (queued or in service), and suppose that a customer that arrives and finds the system full is not lost, but stored in an external queue with infinite space as shown in the figure.



The transition of a customer from the external to the server queue is instantaneous. Therefore, a customer that arrives when the number of customers in the server queue is less than M enters instantaneously the server queue. Similarly, when a customer departs from the server queue, the customer at the head of the external queue moves to the server queue instantaneously. This two- queue system can be modeled as a two dimensional Markov chain with states (i, k) , where $0 \leq i < \infty, 0 \leq k \leq M$.

1. Draw the state transition diagram of the two-dimensional chain.
2. Find the (steady-state) probability $p(i, k)$ that there are i customers in the external queue and k customers in the server queue, $0 \leq i < \infty, 0 \leq k \leq M$
3. Find the average number of customers in the server queue, and the average number of customers in the external queue.
4. Find the average total time that a customer spends in the two-queue system.

3 Solution

3.1 Problem 1

In terms of minutes, $\lambda = \frac{27}{60} = \frac{9}{20}$ per minute.

(a) This is an M/M/1 system, we can use the formula for the total time in the system subtracted by the expected service time.

$$T_q = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{0.9}{0.05} = 18 \text{ minutes}$$

From Little's Law,

$$N_q = \lambda T_q = \frac{9}{20} \cdot 18 = 8.1 \text{ customers}$$

(b) Here, we use the formula for general service process, M/G/1,

$$T_q = \frac{\lambda(E^2[S] + Var[S])}{2(1 - \lambda E[S])} = \frac{(9/20)(4 + 1/3)}{2(1 - 18/20)} = 9.75 \text{ minutes}$$

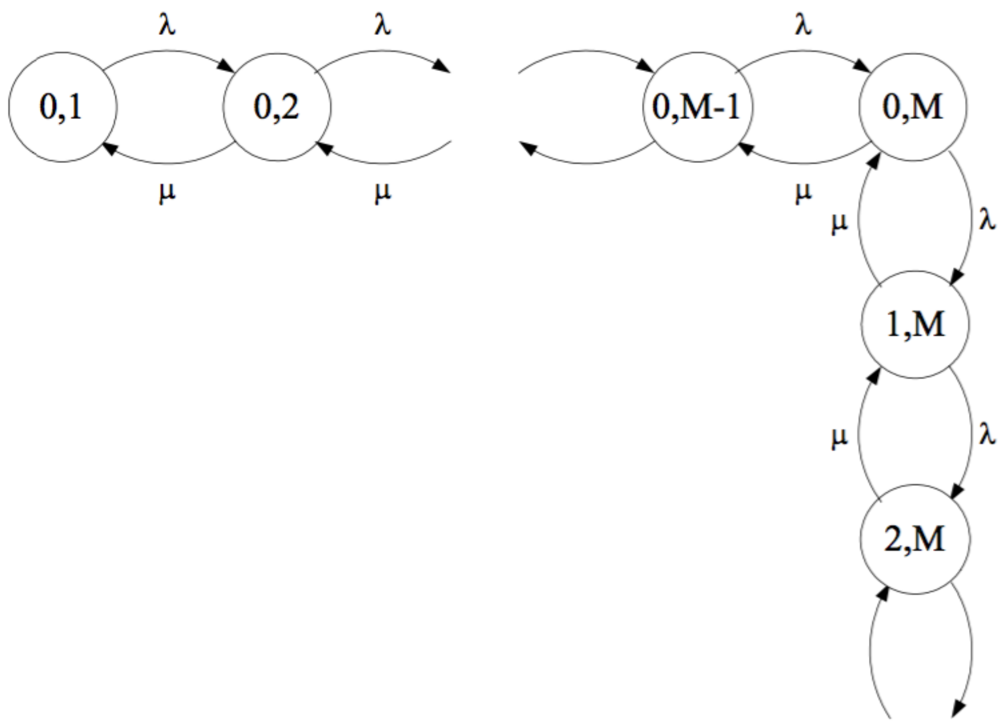
$$N_q = \lambda T_q = \frac{351}{80} = 4.3865 \text{ customers}$$

(c) This is M/D/1 system as the service time is not random. The variance of S is 0 as S is always 2 minutes.

$$T_q = 9 \text{ minutes}, \quad N_q = 4.05 \text{ customers}$$

3.2 Problem 2

1. The state-transition diagram



2. From the diagram, we can get the following system of equations:

$$\mu p(0,k) = \lambda p(0,k-1), \quad k = 1,...,M$$
$$\mu p(i,M) = \lambda p(i-1,M), \quad i = 1,2,...$$
$$\sum_{i=0}^{\infty}\sum_{k=0}^{M} = 1$$

Let $\rho = \lambda/\mu$, we have

$$p(0,k) = \rho^k p(0,0), \quad k = 0,1,...,M$$
$$p(i,M) = \rho^i p(0,M) = \rho^{M+i} p(0,0), \quad i = 0,1,...$$

Then

$$\sum_{i=0}^{\infty}\sum_{k=0}^{M} = 1$$
$$p(0,0)\left(1+\rho+\rho^2+...\right) = 1$$
$$p(0,0)\left(\frac{1}{1-rho}\right) = 1p(0,0) \qquad = 1-\rho$$

3. We first find the marginal distributions for each queue,

$$\begin{aligned} p_1(k) &= p(0,k) = (1-\rho)\rho^k, \quad 0 \le k < M \\ p_1(M) &= \sum_{i=0}^{\infty} p(i,M) = (1-\rho)\rho^M \sum_{i=0}^{\infty} \rho^i = \rho^M \\ p_2(i) &= p(i,M) = (1-\rho)\rho^{M+i}, \quad i \ge 1 \\ p_2(0) &= \sum_{k=0}^{M} p(0,k) = 1 - \rho^{M+1} \end{aligned}$$

Then the average number of customers in the server and external queue are respectively:

$$\begin{aligned} N_1 &= \sum_{k=1}^{M} k p_1(k) = (1-\rho) \sum_{k=1}^{M-1} k\rho^k + M\rho^M = \frac{\rho}{1-\rho}(1-\rho^M) \\ N_2 &= \sum_{i=1}^{\infty} i p_2(i) = (1-\rho)\rho^M \frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}\rho^M \end{aligned}$$

4. The average number of customers in the system is

$$N = N_1 + N_2 = \frac{\rho}{1-\rho}$$

and the average time delay:

$$T = \frac{N}{\lambda} = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu - \lambda}$$