Massachusetts Institute of Technology
1.200J—Transportation Systems Analysis: Performance and Optimization
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**Recitation 5**
**Unit 3 — Probabilistic Methodology and Examples of Applications**

# 1 Negative Exponential Headways

Consider the arrival of busses of MBTA Line 1 (Dudley to Harvard Square) at the stop at 77 Mass Ave in front of MITs main entrance. Assume that the headways between successive arrivals of busses are statistically independent and have a negative exponential distribution with $f_H(x) = \frac{1}{10} \cdot e^{-\frac{x}{10}}$, for $x \geq 0$, with headways measured in minutes.

(a) What is the expected length of the headways, $E[H]$, in minutes?

(b) What is the variance of the headways, $\sigma_h^2$ in minutes?

(c) What is the expected waiting time that a prospective rider of this bus line will experience at the bus stop if she arrives at the bus stop at a random time (relative to the passage of busses)? [This is a case of waiting time under random incidence.]

(d) Suppose a prospective rider arrives at the stop at time t = 0 and is told that the most recent bus to Harvard Square passed 5 minutes ago. What is the probability that this rider will have to wait less than 3 minutes until the next bus arrives?

(e) What is the expected waiting time of this passenger until the next bus arrives? What is the standard deviation of the waiting time?

(f) What is the probability that exactly 3 busses will arrive at the bus stop between t = 0 and t = 40 minutes?

(g) What is the probability that 3 or more busses will arrive at the bus stop between t = 0 and t = 40 minutes?

(h) What is the probability that a total of 6 busses will arrive at the bus stop during the two intervals of [0, 40] minutes and [60, 80] minutes?

Assume now that the arrival at the 77 Mass Ave stop of prospective bus riders to Harvard Square is described by a Poisson process with a rate of $\lambda = 40$ riders per hour. The bus headways are as described above. Suppose that at t = 0, a bus has just left the stop on its way to Harvard Square and there are no riders left waiting at the bus stop.

(i) What is the probability that the next bus to arrive at the stop will find no prospective riders waiting?

(j) What is the probability that the next bus to arrive at the stop will find exactly three prospective riders waiting?

(k) What is the probability that at least three prospective riders will arrive at the bus stop before the next bus to Harvard Square arrives?

(l) Suppose you are told that the first bus after t = 0 arrived at the stop at t = 8 minutes and the second bus arrived at t = 23 minutes. What is the expected number of riders that will be waiting for the first bus? What is it for the second bus? (Assume that the busses have sufficient capacity to pick up all passengers waiting under any circumstances.)

(m) For the situation described in (l), what is the probability of the event that the first bus will find 7 riders waiting and the second bus will find 5 riders?

(n) For the situation described in (l), what is the probability of the event that the first two buses after t = 0, will pick up a total of 12 riders to Harvard Square from the stop?

(o) Without doing any calculations, would you expect the probability of the event described in part (m) to be greater than, equal to, or less than the probability of the event described in part (n)? Please justify your answer in one sentence.

## 1.1 Solution

(a) Because the headways have a negative exponential distribution with parameter $\alpha = \frac{1}{10}$, it follows that $\mathbf{E}[H] = \frac{1}{\alpha} = 10$ minutes.

(b) The variance of the headways, $\sigma_H^2 = \frac{1}{\alpha^2} = 100$ minutes$^2$.

(c) Let $V$ be the waiting time that a prospective rider of this bus line will experience at the bus stop if she arrives at the bus stop at a random time. From lecture 8 page 9, we have

$$\mathbf{E}[V] = \frac{\mathbf{E}[H]}{2} + \frac{\sigma_H^2}{2\mathbf{E}[H]}.$$

Note that $\sigma_H^2 = \mathbf{E}[H]^2$, and thus

$$\begin{aligned}\mathbf{E}[V] &= \mathbf{E}[H] \\ &= 10 \text{ minutes}\end{aligned}$$

(d) Because of the memorylessness of an exponential distribution, the information that the most recent bus passed 5 minutes ago is irrelevant. Let $X$ be the waiting time that this rider will experience. We have that $X$ has a negative exponential distribution with parameter $\alpha = \frac{1}{10}$. Therefore, the probability that this rider will have to wait less than 3 minutes until the next bus arrives is given by

$$\mathbf{P}(X \leq 3) = \int_0^3 f_X(x)dx = \int_0^3 \frac{1}{10}e^{-x/10}dx = 1 - e^{-0.3} = 0.26.$$

Alternatively, the probability that this rider will have to wait less than 3 minutes is also equal to the probability that there is at least one bus arrival in the 3-minute period, which is given by

$$1 - \mathbf{P}(\text{zero arrival in a 3-minute period}) = 1 - P(0,3) = 1 - \frac{(0.3)^0 e^{-0.3}}{0!} = 1 - e^{-0.3} = 0.26$$

(e) As discussed in part (d), it follows that

$$\mathbf{E}[X] = \frac{1}{\alpha} = 10 \text{ minutes, and } \sigma_X = \frac{1}{\alpha} = 10 \text{ minutes.}$$

(f) Because the headways are independent and have a negative exponential distribution, the bus arrival process is a Poisson process with a rate of $\alpha = \frac{1}{10} = 0.1$ buses per minute, or 6 buses per hour. Therefore, the probability that exactly 3 buses will arrive at the bus stop between $t = 0$ and $t = 40$ minutes is given by

$$\frac{(40/10)^3 e^{-40/10}}{3!} = 0.1954.$$

(g) Let $P(k, 40)$ denote the probability that $k$ buses arrive during a 40-minute time interval. $P(k, 40)$ has a Poisson distribution with parameter $40\alpha = 4$. The probability that 3 or more buses will arrive at the bus stop between $t = 0$ and $t = 40$ minutes is given by

$$\begin{aligned}\sum_{k=3}^{\infty} P(k,40) &= 1 - P(0,40) - P(1,40) - P(2,40) \\ &= 1 - \frac{4^0 e^{-4}}{0!} - \frac{4^1 e^{-4}}{1!} - \frac{4^2 e^{-4}}{2!} \\ &= 1 - e^{-4} - 4e^{-4} - 8e^{-4} \\ &= 1 - 13e^{-4} = 0.7619.\end{aligned}$$

(h) Because the number of arrivals during any pre-specificed interval does not depend on the starting time, the probability that a total of 6 buses will arrive at the bus stop during two intervals of $[0, 40]$ minutes and $[60, 80]$ minutes is simply the probability that a total of 6 buses will arrive during an interval of $40 + 20 = 60$ minutes, which is given by

$$P(6,60) = \frac{(60/10)^6 e^{-60/10}}{6!} = 0.1606.$$

(i) Let $X_{bus}$ be the time until the next bus arrival and $X_{rider}$ be the time until the next rider arrival. The bus arrival process is a Poisson process with a rate of 6 buses per hour, and the rider arrival process is a Poisson process with a rate of 40 riders per hour. Therefore, the probability that the next bus to arrive at the stop will find no prospective riders waiting is equal to the probability that the next event is a bus arrival, which is given by

$$\mathbf{P}(X_{bus} < X_{rider}) = \frac{6}{6+40} = 0.1304.$$

(j) **P**(**exactly** three prospective riders waiting) = **P**(the next four events are three rider arrivals followed by a bus arrival) = **P**(1st event is a rider arrival)·**P**(2nd event is a rider arrival)·**P**(3rd event is a rider arrival)·**P**(4th event is a bus arrival) =

$$(\mathbf{P}(X_{bus} > X_{rider}))^3 \mathbf{P}(X_{bus} < X_{rider}) = \left(\frac{40}{6+40}\right)^3 \left(\frac{6}{6+40}\right) = 0.0858.$$

In general, the probability that the next bus finds exactly $k$ prospective rider waiting is given by $\left(\frac{40}{6+40}\right)^k \left(\frac{6}{6+40}\right)$.

(k) **P**(**at least** three prospective riders waiting) = 1 - **P**(no one waiting) - **P**(one rider waiting) - **P**(two riders waiting) =

$$1 - \left(\frac{6}{6+40}\right) - \left(\frac{40}{6+40}\right)\left(\frac{6}{6+40}\right) - \left(\frac{40}{6+40}\right)^2 \left(\frac{6}{6+40}\right) = 0.657$$

Alternatively, the probability that at least three prospective riders will arrive at the bus stop is also equal to the probability that there are three passenger arrivals before a bus arrives (and we do not care what happens after that), which is given by

$$(\mathbf{P}(X_{bus} > X_{rider}))^3 = \left(\frac{40}{6+40}\right)^3 = 0.657$$

(l) Note that the number of riders that will be waiting for the next bus during a $\tau$-hour interval has a Poisson distribution with parameter $40\tau$. Therefore,

$$\mathbf{E}[\text{number of riders waiting for the first bus}] = 40 \cdot \frac{8}{60} = 5.33 \text{ riders, and}$$
$$\mathbf{E}[\text{number of riders waiting for the second bus}] = 40 \cdot \frac{23-8}{60} = 10 \text{ riders.}$$

(m) Using the notation from part (g), we have that the probability of the event that the first bus will find 7 riders waiting **and** the second bus will find 5 riders waiting is given by

$$P(7,8) \cdot P(5,15) = \left[\frac{(40 \cdot 8/60)^7 e^{-40 \cdot 8/60}}{7!}\right] \left[\frac{(40 \cdot 15/60)^5 e^{-40 \cdot 15/60}}{5!}\right] = (0.1176)(0.0378) = 0.0044.$$

(n) The probability of the event that the first two buses will pick up a total of 12 riders is given by

$$P(12,23) = \frac{(40 \cdot 23/60)^{12} e^{-40 \cdot 23/60}}{12!} = 0.0773$$

(o) Without doing any calculation, we know that the probability of the event described in part (m) must be smaller than the probability of the event in part (n) because the event in part (m) is just one special case of the event in part (n). Specifically, the event in part (n) also includes the other combinations of the numbers of riders picked up by the first bus and the second bus.