Massachusetts Institute of Technology
1.200J—Transportation Systems Analysis: Performance and Optimization
Fall 2015 — TA: Wichinpong "Park" Sinchaisri

Recitation 3
Unit 2 — Optimization Methodology

1 Graphical LP Analysis

(a) 2-D

$$\begin{aligned} \underset{x_1,x_2}{\text{Minimize}} \quad & -x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 3, \\ & 2x_1 + x_2 \leq 3, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(b) 3-D

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{Maximize}} \quad & x_1 + x_2 + x_3 \\ \text{subject to} \quad & 0 \leq x_i \leq 1, \ \text{for } i = 1, 2, 3. \end{aligned}$$

(c) Changing Cost Function

Consider the following optimization problem

$$\begin{aligned} \underset{x_1,x_2}{\text{Minimize}} \quad & c_1x_1 + c_2x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Find optimal solutions when the cost vector c is

(i) (1,1)

(ii) (1,0)

(iii) (0,1)

(iv) (-1,-1)

2 Production Example

The start-up *Lab 1.200* just got its first Kickstarter campaign funded, enough to purchase 4,600 units of raw material, and signed a contract with a local manufacturer who can provide up to 5,000 hours of labors. The company is so new that it hasn't got time to properly name its products yet, so they're sadly known by Products 1, 2, 3, and 4. There were exactly 950 Kickstarter backers who requested 1 unit each. Most of them are fine receiving either one of the four products; however, 400 of them specifically requested Product 4.

Resource	Product 1	Product 2	Product 3	Product 4
Raw Material	2	3	4	7
Hours of Labor	3	4	5	6
Sales Price (tens of thousands of $)	4	6	7	8

Table 1: Characteristics of Lab 1.200's products

(a) Formulate an LP that maximizes sales revenue for Lab 1.200.

(b) Solve this optimization in Excel.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D15	Product 1 (units)	0	-1	4	1	1E+30
D16	Product 2 (units)	400	0	6	0.666666667	0.5
D17	Product 3 (units)	150	0	7	1	0.5
D18	Product 4 (units)	400	0	8	2	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D26	raw materials	4600	1	4600	250	150
D27	labor hours	4750	0	5000	1E+30	250
D28	must produce	950	3	950	50	100
D29	P4, at least	400	-2	400	37.5	125

Optimal solution: 6650 (0, 400, 150, 400)

(c) Suppose the price of product 1 is raised by $5,000. What is the new optimal solution to the LP (including the new total sales revenue)?

(d) Suppose the price of product 3 is decreased by $6,000. What is the new optimal solution to the LP?

(e) Suppose that a total of 980 units must be produced. What is the new optimal solution to the LP (including the new total sales revenue)?

(f) Suppose that 4,500 units of raw material are available. What is the new optimal solution to the LP (including the new total sales revenue)? What if only 4,400 units are available?

Solution

Let us define our decision variables as $x_i, i \in \{1, 2, 3, 4\}$, which correspond to the amount of Automobile i to produce. The objective function is as follows:

$$\text{Maximize} \qquad \text{Z} = \sum_{i=1}^{4} s_i x_i$$

where s_i is the sales price of vehicle i.

Constraints exist for raw materials and for hours of labor. In addition, we must produce exactly 950 vehicles, of which at least 400 have to be product 4. As usual, we have a non negativity constraint as well. The formulation of constraints looks like this:

$$\begin{aligned}
2x_1 + 3x_2 + 4x_3 + 7x_4 &\leq 4600 && \text{(Raw Material Constraint)}\\
3x_1 + 4x_2 + 5x_3 + 6x_4 &\leq 5000 && \text{(Labor Constraint)}\\
x_1 + \ \ x_2 + \ \ x_3 + \ \ x_4 &= 950 && \text{(Customer Demand Constraint)}\\
x_4 &\geq 400 && \text{(Product 4 Constraint)}\\
x_i &\geq 0 && \forall i \in i\{1, 2, 3, 4\}
\end{aligned}$$

1. Using Excel Solver, we can find the reduced costs, dual variables (shadow prices) and the allowable increases/decreases. Currently x_1 is not in our basis, so we can calculate the reduced cost of the change in price.

   $$c_i - \alpha_{1i}\pi_1 - \alpha_{2i}\pi_2 - \alpha_{3i}\pi_3 - \alpha_{4i}\pi_4 = 45,000 - 2*10,000 - 3*0 - 1*30,000 - (-20,000)*0 = -5,000$$

   As this is a maximization problem, a variable with a negative reduced cost should not be included in our optimal solution. Alternatively, we see the price increase of \$5,000 is less than the allowable increase of \$10,000. Therefore, there will be no change in the optimal solution and we will continue to produce the following amounts of automobiles: $x = \{0, 400, 150, 400\}$ with sales of \$66.5 million.

2. The price of product 3 (currently in our basis) is decreased by \$6000, which is more than the allowable decrease of \$5,000 seen in the Excel Sensitivity Analysis. We need to therefore re-run excel solver to obtain our new optimal solution, which is now: $x = \{0, 512.5, 0, 437.5\}$ with sales of \$65.75 million.

3. Increasing the number of units produced to 980 is within the allowable increase of 50 units. Therefore we know that we will not introduce a new item into our basis. We are also able to calculate the increase in the objective function using the shadow price of the demand constraint. This is equivalent to $\Delta * \pi = 30 * 30,000 = 900,000$. The new solution is $x = \{0, 520, 60, 400\}$ with sales of \$67.4 million, which is \$900,000 more than our original solution, exactly what we found from the shadow price.

4. Reducing available raw materials to 4500 is within the allowable decrease of the constraint, so we will still produce some quantity x_2, x_3, and x_4 in the optimal solution,

although quantities may change. Reducing the raw materials constraint by 100 units will change the revenue by $\$10,000 * (-100) = -\$1,000,000$. To check this objective value and the new optimal solution, we resolve the problem and find the optimal solution of $x = \{0, 500, 50, 400\}$ with sales of \$65.5 million.

Reducing the materials constraint further to 4400 units means that we will exceed the allowable decrease and need to replace a variable in the optimal solution basis. Resolving the problem using the new constraint value results in an optimal solution of $x = \{50, 500, 0, 400\}$ with sales of \$64 million.