Massachusetts Institute of Technology
1.200J—Transportation Systems Analysis: Performance and Optimization
Fall 2015 — TA: Wichinpong "Park" Sinchaisri

Recitation 12
Quiz 2 Review

1 Queueing Theory

A: Consider a bank with two tellers. Teller 1 has an exponential service time with mean 3 minutes and teller 2's service time is of exponential distribution with rate 10 customers per hour. Menghan, Corey, and Scott enter the bank at the same time. Menghan goes to Teller 1 and Corey goes to Teller 2 while Scott (so nice!) is willing to wait for the first available teller.

(a) What is the expected time that Scott spends in the bank?

(b) What is the expected time until the last of the three customers leaves the bank?

(c) What is the probability that Scott is the last one to leave?

B: Suppose that the number of calls per hour to a virtual TA service follows a Poisson process with rate 4/hour.

(a) What is the probability that fewer than 2 calls arrived in the first hour?

(b) Suppose that 6 calls arrived in the first hour, what is the probability that there will be fewer than 2 calls in the second hour?

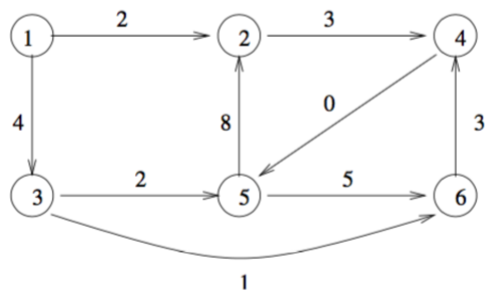
(c) Given that 6 calls arrived in the first two hours, what is the probability that exactly 2 arrived in the first hour and 4 in the second hour?

C: Consider a taxi station at Boston Logan Airport where taxis and customers arrive with respect to Poisson process of rates 2/min and 3/min, respectively. Suppose that a taxi will wait no matter how many other taxis are present. However, if an arriving person does not find a taxi waiting he leaves to find an alternative transportation (such as donkeys and horses).

(a) What is the long run probability that an arriving customer gets a taxi?

(b) What is the average number of taxi waiting?

2 Network Flows



(a) Formulate an LP that find the shortest path from vertex 1 to vertex 6.

(b) Fibonacci Heap: findMin and deleteMin takes $O(\log n)$, update takes $O(1)$, what's the running time for Fibonacci Heap-based Dijkstra?

3 Simulation

(a) Describe a pseudo-code to generate the following random variables:

- X taking values in [0,1], having p.d.f.

$$f(x) = e^x/(e-1)$$

- Y from the c.d.f:

$$F(y) = \frac{y + y^3 + y^5}{3}, 0 \leq y \leq 1$$

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(b) Discrete Event Simulation: Be sure to describe:

- Set of states: $\{0, 1, 2, ...\} \times \{0, 1\}$ forms a pair (n, i) to represent number of jobs (waiting or under service) and i whether Park takes a break.
- State transitions: in words and in states. For example, a new job arrives, whether the server is present or not: $(n, i) \rightarrow (n + 1, i)$.
- Event times: time of new job arrival, time that job is completed
- Counter variables: what to keep track of: indices for job arrival/completion number
- Input variables: what needs to be given: arrival rate, service time rate
- Output variables: what we want to collect: arrival time of job i, completion time of job i.
- General organization of the program: Usually a `while` loop that keeps running until the termination condition is met.

Thanks for a fun semester! Good Luck! :)

4 Solution

4.1 Queues

A:

(a) Let S_1, S_2 be the service time of Teller 1 and Teller 2, respectively.

$$\begin{aligned}
\mathbb{E}[\text{time Scott spends}] &= \mathbb{E}[\min(S_1,S_2)] + P(S_1 > S_2)\mathbb{E}[S_2] + P(S_1 < S_2)\mathbb{E}[S_1] \\
&= \frac{1}{1/3+1/6} + \frac{1/6}{1/3+1/6}\cdot 6 + \frac{1/3}{1/3+1/6}\cdot 3 \\
&= 2+2+2 = 6 \text{ minutes}
\end{aligned}$$

Note that we use the property that the minimum of exponential random variables is also an exponential random variable with parameter being the combined parameters of the set. Thus, $\min(S_1,S_2) \sim \mathit{Expo}(1/3+1/6)$, so the expected value of $\min(S_1,S_2) = 1/(1/3+1/6)$.

Proof:

$$\begin{aligned}
P(\min(S_1,S_2) > t) &= P(S_1 > t, S_2 > t) \\
&= P(S_1 > t)P(S_2 > t) \\
&= (1 - P(S_1 \le t))(1 - P(S_2 \le t)) \\
&= e^{-(1/3)t}e^{-(1/6)t} \\
&= e^{-(1/3+1/6)t}
\end{aligned}$$

Thus, $P(\min(S_1,S_2) \le t) = 1 - e^{-(1/3+1/6)t}$ shows that the minimum between the two has the same cumulative distribution with an exponential random variable with parameter $1/3+1/6$.

(b) Because the customer to finish last can be Scott or either Corey or Menghan, we look at two terms. We first look at the minimum time between the two because that is the time Scott waits before starting the service. Then, once he starts, we look at the maximum between the two as that will determine the time the last out of the three customers finishes.

First, we notice that

$$\mathbb{E}[S_1+S_2] = \mathbb{E}[\min(S_1,S_2)+\max(S_1,S_2)] = \mathbb{E}[\min(S_1,S_2)] + \mathbb{E}[\max(S_1,S_2)]$$

$$\begin{aligned}
\mathbb{E}[\text{time until last customer finishes}] &= \mathbb{E}[\min(S_1,S_2)] + \mathbb{E}[\max(S_1,S_2)] \\
&= \mathbb{E}[\min(S_1,S_2)] + \mathbb{E}[S_1+S_2] - \mathbb{E}[\min(S_1,S_2)] \\
&= \mathbb{E}[S_1] + \mathbb{E}[S_2] \\
&= 3+6 = 9 \text{ minutes}
\end{aligned}$$

(c) The probability that Scott is the last one to leave:

$$P(S_1 < S_2)P(S_2 < S_1) + P(S_2 < S_1)P(S_1 < S_2) = 2\cdot\frac{1}{3}\frac{2}{3} = \frac{4}{9}$$

B:

(a) Using the fact that the number of events with exponentially distributed inter-arrival time is a Poisson process.

$$P(N(1) < 2) = P(N(1) = 0) + P(N(1) = 1) = e^{-4} + 4e^{-4} = 5e^{-4}$$

(b) Because Poisson process is memory less, the given information about the past does not affect the current/future states.

$$P(N(2) - N(1) < 2 | N(1) = 6) = P(N(1) < 2) = 5e^{-4}$$

(c) This is different than the previous question as the given information DOES affect the current state.

$$\begin{aligned} P(N(1) = 2, N(2) - N(1) = 4 | N(2) = 6) &= \frac{P(N(1) = 2) P(N(2) - N(1) = 4)}{P(N(2) = 6)} \\ &= \frac{e^{-4} 4^2/2! \cdot e^{-4} 4^4/4!}{e^{-8} 8^6/6!} \end{aligned}$$

Notice that this probability is greater than just $P(N(1) = 2, N(2) - N(1) = 4)$ because we are certain that 6 calls did arrive in the 2-hour interval.

C:

(a) Let $N(t)$ be the number of taxis waiting at time t. Then $\{N(t), t \geq 0\}$ is a M/M/1, with arrival rate 2 and service rate 3. An arriving customer gets a taxi with probability $1 - P_0 = 1 - (1 - \rho) = 1 - (1 - 2/3) = 2/3$.

(b) We know that $P_n = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n$. Thus,

$$\sum_{n=0}^{\infty} n \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n = 2$$

4.2 Network Flows

(a)

$$\begin{aligned} \underset{x_{ij}}{\text{Minimize}} \quad & \sum_{(i,j) \in A} w_{ij} x_{ij} = 2x_{12} + 4x_{13} + 3x_{24} + 2x_{35} + x_{36} + 8x_{52} + 5x_{56} + 3x_{64} \\ \text{subject to} \quad & \sum_j x_{ij} - \sum_j x_{ji} = 1, \quad \text{if } i = 1; \\ & \sum_j x_{ij} - \sum_j x_{ji} = -1, \quad \text{if } i = 6; \\ & \sum_j x_{ij} - \sum_j x_{ji} = 0, \quad \text{otherwise}; \\ & x_{ij} \geq 0. \end{aligned}$$

(b) From $O(m \cdot t_{update} + n \cdot (t_{findMin} + t_{deleteMin})$, we got $O(m + n \log n)$

4.3 Simulation

(a) CDF is

$$F(x) = \int_0^x f(s)ds = \frac{e^x - 1}{e - 1}$$

for $0 \leq x \leq 1$. The function $u = F(x)$ is easily inverted: $x = G(u)$ where

$$G(u) = \log(1 + (e - 1)u).$$

The pseudo-code: `m = 100000;`
Generate a random variable U: `U = rand(1,m);`

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X = log(1 + (exp(1)-1)*U);
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(b) Let $F_1(y) = y, F_2(y) = y^3, F_3(y) = y^5$. Note that $y = F_i^{-1}(u) = u^{1/(2i-1)}$ for $i = 1, 2, 3$. With this in mind, we can describe an algorithm for generating a random variable Y with distribution function $F(y)$ as follows:

i) Generate a random number U uniformly distributed over [0, 1];

ii) Let $I = \lfloor(3U)\rfloor + 1$; (this produces a random index from $\{1, 2, 3\}$with probabilities 1/3.)

iii) Generate another random number V independent of U, also uniform over [0, 1];

iv) Now set $Y = V^{1/(2I-1)}$.