1 Cumulative Diagrams

Consider the following figure. It shows the demand rate and the capacity of a roadway. Cars generally access the road at a rate $\lambda$. As is typical during rush hour, there is an increased rate of people using the roadway. In this example, this higher rate is $1.5\lambda$ and occurs from time $t = a$ to $t = b$. From time $t = b$ to $t = c$, an accident reduces the capacity to $0.5\mu$.
(a) Draw the Cumulative Diagram.

The other diagram that can be useful for calculations is the length of queue versus time.
(b) At what time does the traffic clear?

**Solution** Looking both diagrams, we can find the time \( t^* \) that the arrivals \( (A(t)) \) and departures \( (D(t)) \) meet, or when the length of queue hits zero. From the length of queue diagram:

\[
(t^* - c)(\mu - \lambda) = (1.5\lambda - \mu)(b - a) + (\lambda - 0.5\mu)(c - b)
\]

\[
t^* = \frac{(1.5\lambda - \mu)(b - a) + (\lambda - 0.5\mu)(c - b)}{\mu - \lambda} + c
\]

(c) What is the total delay experienced by all vehicles that experience delay?

**Solution** The total delay is the area between \( A(t) \) and \( D(t) \) in the cumulative diagram or the area under the length of queue diagram, which is

\[
\frac{1}{2}(b-a)^2(1.5\lambda - \mu) + \frac{1}{2}(c-b)(2(1.5\lambda - \mu)(b-a) + (\lambda - 0.5\mu)(c-b)) + \frac{1}{2}(t^*-c)((1.5\lambda - \mu)(b-a) + (\lambda - 0.5\mu)(c-b)).
\]

(d) How many vehicles experience delay?

**Solution** We can find this by computing how many vehicles arrive during the period that there is a queue. That is, adding up the rate of arrival multiplied by the length of interval for each time interval:

\[
(b - a)(1.5\lambda) + (t^* - b)\lambda
\]

(e) Which vehicle has the longest delay? How long is the delay?

**Solution** The two candidates we should compare are 1) the one that arrives at time \( b \) and 2) the one that departs the queue at time \( c \). The first candidate is the \( a\lambda + 1.5(b - a)\lambda \)-th vehicle who waits \( \frac{\mu}{2(1.5\lambda - \mu)(b-a)} \). (We assume that \( 2(1.5\lambda - \mu)/\mu < (c - b)/(b - a) \).) The second candidate departing at time \( c \) is the \( (a\lambda + \mu(b - a) + (c - b)(\mu/2)) \)-th vehicle. We can find his arrival time by looking at \( A(t) \) and find out the x-axis \( (t) \) value when \( A(t) = a\lambda + \mu(b - a) + (c - b)(\mu/2) \).

## 2 Runway Capacity at King’s Landing

King’s Landing has one small operating airport which could only serve medium (M) and light (L) aircraft until this Fall where heavy (H) aircraft like Boeing 787 can be routed here. Under ICAO’s rules, the set of separations for arrivals on final approach are as given below:

<table>
<thead>
<tr>
<th>Separation (nmiles)</th>
<th>Trailing Aircraft</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>6</td>
</tr>
<tr>
<td>M</td>
<td>4</td>
</tr>
<tr>
<td>L</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1: Minimum separations for arrivals on final approach

Consider now a runway used only for arrivals at a major airport where ICAO separation standards are in use. Assume that the traffic mix at that airport and the aircraft characteristics are as follows:

The length of the final approach path, \( r \), is 7 nautical miles.

Compute the arrivals capacity of this runway when \( x = 10 \), and compare with the capacity prior to the introduction of the heavy aircraft.
### Solution
Two main components: the joint probability of two arrivals and the separation time between successive aircraft. The following table shows the minimum time separation between aircraft, which will be constant across the three scenarios.

\[
T_{ij} = \max \left[ \frac{r + s_{ij}}{v_j} - \frac{r}{v_i}, o_i \right], v_i > v_j \\
T_{ij} = \max \left[ \frac{s_{ij}}{v_j}, o_i \right], v_i \leq v_j
\]

Don’t forget to convert between seconds and hours!

<table>
<thead>
<tr>
<th>Aircraft Class</th>
<th>Approach Speed (knots)</th>
<th>Mix (%)</th>
<th>Runway Occupancy Time (ROT) on Landing (seconds)</th>
<th>Seats</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>150</td>
<td>$x$</td>
<td>70</td>
<td>320</td>
</tr>
<tr>
<td>M</td>
<td>135</td>
<td>$70 - x$</td>
<td>60</td>
<td>140</td>
</tr>
<tr>
<td>L</td>
<td>105</td>
<td>30</td>
<td>50</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 2: Aircraft characteristics

Next, we calculate the probability of type $i$ aircraft followed by type $j$ aircraft for all $i$ and $j$ and each mix of H’s.

<table>
<thead>
<tr>
<th>Aircraft Class</th>
<th>H</th>
<th>M</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M</td>
<td>0</td>
<td>0.49</td>
<td>0.21</td>
</tr>
<tr>
<td>L</td>
<td>0</td>
<td>0.21</td>
<td>0.49</td>
</tr>
</tbody>
</table>

Table 4: Joint probability, $x = 0$

<table>
<thead>
<tr>
<th>Aircraft Class</th>
<th>H</th>
<th>M</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>0.01</td>
<td>0.06</td>
<td>0.03</td>
</tr>
<tr>
<td>M</td>
<td>0.06</td>
<td>0.36</td>
<td>0.18</td>
</tr>
<tr>
<td>L</td>
<td>0.03</td>
<td>0.18</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table 5: Joint probability, $x = 10$

From here, we can find the expected value of minimum inter-arrival time, which is equal to

\[
E[T] = \sum_{i}^{K} \sum_{j}^{K} p_{ij} T_{ij}
\]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$E[T]$</th>
<th>Arrivals/hr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>150.3543</td>
<td>23.94 → 24</td>
</tr>
<tr>
<td>10</td>
<td>145.7905</td>
<td>24.69 → 25</td>
</tr>
</tbody>
</table>

Table 6: Expected value of inter-arrival times and arrival capacities
3 Traffic Flow Models

Greenshield  The basic relationship for the Greenshield model: \( v = v_{\text{max}} \left( 1 - \frac{k}{k_{\text{jam}}} \right) \).

Suppose \( v_{\text{max}} = 48 \text{ km/h} \) and \( k_{\text{jam}} = 125 \text{ veh/km} \). Plot the amount of time \( t \) it will take to travel these 3 km as a function of the flow \( q \) on the road segment.

**Solution**  Because \( q = vk \), we get
\[
q = v_{\text{max}}k \left( 1 - \frac{k}{k_{\text{jam}}} \right)
\]

Taking the derivative \( dq/dk = 0 \), we can find the critical density that gives us \( q_{\text{max}} \). The critical density \( k_c = k_{\text{jam}}/2 \). Substituting this into the fundamental relationship, we can find the value of \( v \) associated with \( k_c \).

Extended Free Flow  Let \( k_j \) be the jam density of a single lane of a highway. The \( v \) vs. \( k \) relation is given as follows:

\[
v = \begin{cases} 
  v_f, & \text{if } k \leq k^*, \\
  c \left( \frac{k_j}{k} - 1 \right), & \text{if } k \geq k^*
\end{cases}
\]

where \( v \) is in units of km/hour and \( k^* \) (the critical density), \( v_f \) (the “free flow speed”), \( c \) and \( k_j \) are all constants. Note that \( v \) has a single value at \( k = k^* \), i.e., (1a) and (1b) give the same value of \( v \) when \( k = k^* \).

We shall now consider the East-to-West (EW) lane of a two-lane rural highway. It has been observed that the EFF model provides a good approximation to the traffic flow characteristics of the EW lane. It has also been determined that, in this case, \( q_{\text{max}} = 1600 \text{ vehicles per hour} \) (this is the maximum flow that the lane can support), \( v_f = 80 \text{ km/hour} \) and \( c = 20 \).

(a) Please draw the \( q \) vs. \( k \) diagram for this lane of the rural highway.

**Solution**  The fundamental relationship \( (k - q) \) for the lane of the rural highway:

For \( 0 \leq k \leq k^* \),
\[
q = ku = kuf = 80k
\]
For \( k \geq k^* \),

\[
q = ku = kc \left( \frac{k_j}{k} - 1 \right) = 20(k_j - k)
\]

The relationship is continuous, then we can find \( q_{\text{max}} = 80k^* \). Given the value of \( q_{\text{max}} \), we obtain \( k^* = 1600/80 = 20 \). We are certain that at \( k = k^* \) gives the maximum \( q \) because \( u \) is a decreasing function in \( k \) after \( k = k^* \). We can then find that \( 1600 = 20(k_j - 20) \). Therefore, \( k_j = 100 \) vehicles/km.

(b) Consider a 2-kilometer stretch of this highway. Plot the \( t \) vs. \( q \) relationship for a car traveling on one of the lanes of this highway for all the possible values of \( q \). In other words, show on the vertical axis the amount of time it will take to travel this 2-kilometer segment for all possible values of \( q \). Is \( t \) a single-valued function of \( q \)?

**Solution** To determine the plot of time compared to the flow, we first derive the density as a function of speed. We also plug in the formula \( u = \frac{d}{t} \) to get density as a function of time. We then plug this into \( q = uk \) to get flow as a function of time.

The relationship between \( t \) and \( q \) can be found from:

\[
t = \frac{\text{distance}}{u} = \frac{2}{q/k} = \frac{2k}{q}
\]

For \( 0 \leq k \leq k^* \):

\[
t = \frac{2k}{u_f k} = \frac{2}{u_f} = 0.025.
\]

For \( k^* \leq k \leq k_j \):

\[
u = \frac{\text{distance}}{\text{time}} = 20 \left( \frac{100}{k} - 1 \right)
\]

\[
\frac{2}{t} = 20 \left( \frac{100 - k}{k} \right)
\]

\[
k = 10t(100 - k)
\]

\[
k(1 + 10t) = 1000t
\]

\[
k = \frac{1000t}{10t + 1}
\]

\[
q = uk = 2000 - 20k
\]

\[
q = 2000 - \frac{20000t}{10t + 1}
\]

\[
t = \frac{200}{q} = \frac{1}{10}
\]
$t$ is not a single-valued function of $q$ (as we have seen from the relationship between them and the figures above) because the same flow rate $q$ can be achieved at two different times $t$ and densities $k$. 